MODULES WITH REDUCIBLE COMPLEXITY

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Abstract. For a commutative Noetherian local ring we define and study the class of modules having reducible complexity, a class containing all modules of finite complete intersection dimension. Various properties of this class of modules are given, together with results on the vanishing of homology and cohomology.

1. Introduction

The complexity of a finitely generated module over a commutative Noetherian local ring is a measure on a polynomial scale of the growth of the ranks of the free modules in its minimal free resolution. Over a local complete intersection every finitely generated module has finite complexity. In fact, it follows from [Gul, Theorem 2.3] that this property characterizes a local complete intersection, and that it is equivalent to the complexity of the residue field being finite.

In [AGP] a certain finiteness condition was defined under which a module *behaves* homologically like a module over a complete intersection. Namely, a module M over a commutative Noetherian local ring A has finite *complete intersection dimension* if there exist local rings R and Q and a diagram $A \to R \twoheadleftarrow Q$ of local homomorphisms (called a *quasi-deformation* of A) such that $A \to R$ is faithfully flat, $R \twoheadleftarrow Q$ is surjective with kernel generated by a regular sequence, and $\operatorname{pd}_Q(R \otimes_A M)$ is finite. There is of course a reason behind the choice of terminology; it was shown in [AGP] that a local ring is a complete intersection if and only if all its finitely generated modules have finite complete intersection dimension, and that this is equivalent to the finiteness of the complete intersection dimension of the residue field. Moreover, it was shown that if the projective dimension of a module is finite, then it is equal to the complete intersection dimension of the module.

We shall study a class of modules whose complexity is "reducible" (defined in the next section), a class which contains all modules of finite complete intersection dimension. In particular, we investigate for this class of modules the vanishing of homology and cohomology, and generalize several of the results in [ArY], [ChI], [Jo1] and [Jo2].

2. reducible complexity

Throughout this paper we let $(A, \mathfrak{m}, k)$ be a commutative Noetherian local ring, and we suppose all modules are finitely generated. We fix a finitely generated nonzero A-module M with minimal free resolution

$$\mathbf{F}_M \colon \cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

The rank of F_n, i.e. the integer $\dim_k \operatorname{Ext}_A^n(M,k)$, is the n'th *Betti number* of M, and we denote this by $\beta_n(M)$. The *complexity* of M, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M = \inf\{t \in \mathbb{N}_0 \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq a n^{t-1} \text{ for } n \gg 0\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In general the complexity of a module may be infinite, whereas it is zero if and only if the module has finite projective dimension.

We now give the main definition, which is inductive and defines the class of modules having reducible complexity as a subcategory of the category of (finitely generated) A-modules having finite complexity. However, before stating the definition, recall that the n'th *syzygy* of an A-module X with minimal free resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0$$

is the cokernel of $P_{n+1} \to P_n$ and denoted by $\Omega_A^n(X)$ (note that $\Omega_A^0(X) = X$), and it is unique up to isomorphism. For an A-module Y and a homogeneous element $\eta \in \operatorname{Ext}_A^*(X, Y)$, choose a map $f_\eta \colon \Omega_A^{|\eta|}(X) \to Y$ representing η, and denote by K_η the pushout of this map and the inclusion $\Omega_A^{|\eta|}(X) \hookrightarrow P_{|\eta|-1}$. As parallel maps in a pushout diagram have isomorphic cokernels, we obtain an exact sequence

$$0 \to Y \to K_\eta \to \Omega_A^{|\eta|-1}(X) \to 0.$$

Note that the middle term K_η is independent (up to isomorphism) of the map f_η chosen to represent η.

Definition 2.1. *Let $\mathcal{C}_A$ denote the category of all A-modules having finite complexity. The subcategory $\mathcal{C}_A^r \subseteq \mathcal{C}_A$ of modules having* reducible complexity *is defined inductively as follows:*

(i) *Every module of finite projective dimension belongs to $\mathcal{C}_A^r$.*
(ii) *A module $X \in \mathcal{C}_A$ with $\operatorname{cx} X > 0$ belongs to $\mathcal{C}_A^r$ if there exists a homogeneous element $\eta \in \operatorname{Ext}_A^*(X, X)$ of positive degree such that $\operatorname{cx} K_\eta < \operatorname{cx} X, \operatorname{depth} K_\eta = \operatorname{depth} X$ and $K_\eta \in \mathcal{C}_A^r$. We say that the element η* reduces the complexity *of M.*

Remark. The condition $\operatorname{depth} K_\eta = \operatorname{depth} X$ is not very strong; suppose for example that $\operatorname{depth} \Omega_A^i(X) \leq \operatorname{depth} A$ for all i (this always happens when A is Cohen-Macaulay). Then we must have $\operatorname{depth} X \leq \operatorname{depth} \Omega_A^{|\eta|-1}(X)$, implying the equality $\operatorname{depth} K_\eta = \operatorname{depth} X$.

Note the trivial fact that if every A-module has reducible complexity, then A must be a complete intersection since then by definition every module has finite complexity. The following result shows that the converse is also true, in fact every module of finite complete intersection dimension has reducible complexity. Moreover, if A is Cohen-Macaulay and M has reducible complexity, then so does any syzygy of M.

Proposition 2.2. (i) *Every module of finite complete intersection dimension has reducible complexity.*
(ii) *If A is Cohen-Macaulay and M has reducible complexity, then so does the kernel of any surjective map $F \twoheadrightarrow M$ when F is free. In particular, any syzygy of M has reducible complexity.*
(iii) *if B is a local ring and $A \to B$ a faithfully flat local homomorphism, then if M has reducible complexity, so does the B-module $B \otimes_A M$.*

Proof. (i) If M has finite complete intersection dimension, then from [AGP, Proposition 5.2] it follows that the complexity of M is finite. We argue by induction on $\operatorname{cx} M$ that M has reducible complexity, the case $\operatorname{cx} M = 0$ following from the definition. Suppose therefore that the complexity of M is nonzero. By [AGP, Proposition 7.2] there exists an eventually surjective chain map of degree $-n$ (where $n > 0$) on the minimal free resolution of M. This chain map corresponds to an element η of

$\operatorname{Ext}_A^n(M,M)$, giving the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^n(M) \arrow[r] \arrow[d, "f_\eta"] & F_{n-1} \arrow[r] \arrow[d] & \Omega_A^{n-1}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & K_\eta \arrow[r] & \Omega_A^{n-1}(M) \arrow[r] & 0
\end{tikzcd}$$

of A-modules.

Now consider the exact sequence involving K_η. Since M and $\Omega_A^{n-1}(M)$ both have finite complete intersection dimension, then so does K_η. The exact sequence gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}_A^i(K_\eta,k) \to \operatorname{Ext}_A^i(M,k) \xrightarrow{\partial_\eta} \operatorname{Ext}_A^{i+n}(M,k) \to \operatorname{Ext}_A^{i+1}(K_\eta,k) \to \cdots,$$

where $\operatorname{Ext}_A^i(\Omega_A^{n-1}(M),k)$ has been identified with $\operatorname{Ext}_A^{i+n-1}(M,k)$. It follows from [Mac, Theorem III.9.1] that the connecting homomorphism $\partial_\eta \colon \operatorname{Ext}_A^i(M,k) \to \operatorname{Ext}_A^{i+n}(M,k)$ is then scalar multiplication by $(-1)^i\eta$ (we think of $\operatorname{Ext}_A^*(M,k)$ as a graded right module over the graded ring $\operatorname{Ext}_A^*(M,M)$), and reversing the arguments in the proof of [Ber, Proposition 2.2] shows that this is an injective map for $i \gg 0$. Consequently the equality $\beta_{i+1}(K_\eta) = \beta_{i+n}(M) - \beta_i(M)$ holds for large values of i. By [AGP, Theorem 5.3] the complexities $\operatorname{cx} M$ and $\operatorname{cx} K_\eta$ equal the orders of the poles at $t=1$ of the Poincaré series $\sum \beta_n(M)t^n$ and $\sum \beta_n(K_\eta)t^n$, respectively, thus $\operatorname{cx} K_\eta = \operatorname{cx} M - 1$.

It remains only to show that $\operatorname{depth} K_\eta = \operatorname{depth} M$, but this is easy; from [AGP, Theorem 1.4] we have $0 \leq \text{CI-dim}\, \Omega_A^i(M) = \operatorname{depth} A - \operatorname{depth} \Omega_A^i(M)$ for each $i \geq 0$, and this implies the inequality $\operatorname{depth} \Omega_A^i(M) \leq \operatorname{depth} \Omega_A^{i+1}(M)$. In particular we have $\operatorname{depth} M \leq \operatorname{depth} \Omega_A^{n-1}(M)$, and therefore $\operatorname{depth} K_\eta$ must equal $\operatorname{depth} M$.

(ii) Let L denote the kernel of the surjective map $F \twoheadrightarrow M$. Again we argue by induction on $\operatorname{cx} M$. If the projective dimension of M is finite, then so is the projective dimension of L, and we are done. Suppose therefore $\operatorname{cx} M$ is nonzero, and let $\eta \in \operatorname{Ext}_A^*(M,M)$ be an element reducing the complexity of M. By the Horseshoe Lemma we have an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & K \oplus Q \arrow[r] \arrow[d] & \Omega_A^{|\eta|-1}(L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F' \oplus Q \arrow[r] \arrow[d] & F'' \oplus Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & K_\eta \arrow[r] \arrow[d] & \Omega_A^{|\eta|-1}(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where F' and F'' are free modules, and Q is a free module such that $\Omega_A^{|\eta|}(M) \oplus Q \simeq \Omega_A^{|\eta|-1}(L)$. If M and K_η are maximal Cohen-Macaulay, then so are L and $K \oplus Q$, and if not then $\operatorname{depth} L = \operatorname{depth} M + 1$ and $\operatorname{depth}(K \oplus Q) = \operatorname{depth} K_\eta + 1$. In either case we see that $\operatorname{depth} L$ equals $\operatorname{depth}(K \oplus Q)$. Moreover, we have $\operatorname{cx}(K \oplus Q) = \operatorname{cx} K_\eta < \operatorname{cx} M = \operatorname{cx} L$, and so by induction we are done.

Note that the upper horizontal short exact sequence in the above diagram corresponds to an element θ in $\operatorname{Ext}_A^{|\eta|}(L,L)$ reducing the complexity of L. A map $f_\theta \colon \Omega_A^{|\eta|}(L) \to L$ representing θ is obtained by lifting a map f_η representing η along the minimal free resolution of M, as in the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_A^{|\eta|+1}(M) \arrow[r] \arrow[d, dotted, "\Omega_A(f_\eta)"] & F_{|\eta|} \arrow[r] \arrow[d, dotted] & \Omega_A^{|\eta|}(M) \arrow[r] \arrow[d, "f_\eta"] & 0 \\ 0 \arrow[r] & L \arrow[r] & F \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

In this way we obtain a map $\Omega_A(f_\eta) \colon \Omega_A^{|\eta|+1}(M) \to L$, and adding to $\Omega_A^{|\eta|+1}(M)$ a free module Q' such that $\Omega_A^{|\eta|+1}(M) \oplus Q' \simeq \Omega_A^{|\eta|}(L)$, we obtain a map $\Omega_A^{|\eta|}(L) \to L$ representing θ.

(iii) If X is any A-module with a minimal free resolution $\mathbf{F}_X$, then the complex $B \otimes_A \mathbf{F}_X$ is a minimal free B-resolution of $B \otimes_A X$, giving the equality $\operatorname{cx}_A X = \operatorname{cx}_B(B \otimes_A X)$. Moreover, if Y is another A-module then from [Mat, Theorem 23.3] we have $\operatorname{depth}_A X - \operatorname{depth}_A Y = \operatorname{depth}_B(B \otimes_A X) - \operatorname{depth}_B(B \otimes_A Y)$. Hence if $\eta \in \operatorname{Ext}_A^*(M,M)$ reduces the complexity of M, the element $B \otimes_A \eta \in \operatorname{Ext}_B^*(B \otimes_A M, B \otimes_A M)$ reduces the complexity of $B \otimes_A M$. $\square$

Thus the class of modules having finite complete intersection dimension is contained in the class of modules having reducible complexity. However, the following example shows that the inclusion is strict in general; there are a lot of modules having reducible complexity but whose complete intersection dimension is infinite.

Example. Suppose M is periodic of period $p \geq 3$ (i.e. $\Omega_A^p(M)$ is isomorphic to M), and that $\operatorname{depth} M \leq \operatorname{depth} \Omega_A^{p-1}(M)$. Then we have an exact sequence

$$0 \to M \to F_{p-1} \to \Omega_A^{p-1}(M) \to 0,$$

and we have $0 = \operatorname{cx} F_{p-1} = \operatorname{cx} M - 1$ and $\operatorname{depth} F_{p-1} = \operatorname{depth} M$. Therefore M has reducible complexity, and it cannot be of finite complete intersection dimension since then by [AGP, Theorem 7.3] the period would have been two.

An example of such a module was given in [GaP, Section3]; let $(A, \mathfrak{m}, k)$ be the commutative local finite dimensional k-algebra $k[x_1,x_2,x_3,x_4]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$x_1^2, \quad x_2^2, \quad x_3^2, \quad x_4^2, \quad x_3x_4, \quad x_1x_4 + x_2x_4, \quad \alpha x_1x_3 + x_2x_3$$

for a nonzero element $\alpha \in k$. The complex

$$\cdots \to A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_0} M \to 0,$$

where the maps are given by the matrices

$$d_n = \begin{pmatrix} x_1 & \alpha^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix},$$

is a minimal free resolution of the module $M := \operatorname{Im} d_0$, and so if α has finite order p we see that M is periodic of period p.

It is worth mentioning that there do exist examples of modules over Gorenstein rings whose complete intersection dimension is not finite but which have reducible complexity (see [GaP, Proposition 3.1] for an example similar to that above).

Now let X and Y be arbitrary A-modules, and $\theta_1 \in \operatorname{Ext}_A^*(X,X)$ and $\theta_2 \in \operatorname{Ext}_A^*(X,Y)$ two homogeneous elements. The following lemma, motivated by [EHSST, Lemma 7.2], links K_{θ_1} and K_{θ_2} to $K_{\theta_2\theta_1}$, and will be a key ingredient in several of the forthcoming results.

Lemma 2.3. *If $\theta_1 \in \operatorname{Ext}^*_A(X,X)$ and $\theta_2 \in \operatorname{Ext}^*_A(X,Y)$ are two homogeneous elements, then there exists an exact sequence*

$$0 \to \Omega_A^{|\theta_2|}(K_{\theta_1}) \to K_{\theta_2\theta_1} \oplus F \to K_{\theta_2} \to 0$$

of A-modules, where F is free.

Proof. Denote $|\theta_i|$ by n_i for $i = 1, 2$. The element θ_1 gives rise to an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^{n_1}(X) \arrow[r, "i"] \arrow[d, "f_{\theta_1}"] & Q_{n_1-1} \arrow[r, "\pi"] \arrow[d, "h"] & \Omega_A^{n_1-1}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "j"] & K_{\theta_1} \arrow[r] & \Omega_A^{n_1-1}(X) \arrow[r] & 0
\end{tikzcd}$$

where Q_n denotes the n'th module in the minimal free resolution of X. Letting $Q \xrightarrow{g} X$ be a surjection, where Q is free, we can modify the diagram and obtain

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{Ker}(h, jg) \arrow[r, equal] \arrow[d] & \operatorname{Ker}(h, jg) \arrow[d] & & \\
0 \arrow[r] & \Omega_A^{n_1}(X) \oplus Q \arrow[r, "\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "(f_{\theta_1}, g)"] & Q_{n_1-1} \oplus Q \arrow[r, "(\pi, 0)"] \arrow[d, "(h, jg)"] & \Omega_A^{n_1-1}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "j"] \arrow[d] & K_{\theta_1} \arrow[r] \arrow[d] & \Omega_A^{n_1-1}(X) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\operatorname{Ker}(h, jg)$ is isomorphic to $\Omega_A^1(K_{\theta_1}) \oplus Q'$ for some free module Q', the left vertical exact sequence yields an exact sequence

$$0 \to \Omega_A^1(K_{\theta_1}) \oplus Q' \to \Omega_A^{n_1}(X) \oplus Q \xrightarrow{(f_{\theta_1}, g)} X \to 0$$

on which we can apply the Horseshoe Lemma and obtain an exact sequence

$$\mu \colon 0 \to \Omega_A^{n_2}(K_{\theta_1}) \to \Omega_A^{n_1+n_2-1}(X) \oplus F \xrightarrow{(\Omega_A^{n_2-1}(f_{\theta_1}), s)} \Omega_A^{n_2-1}(X) \to 0,$$

where F is free and $F \xrightarrow{s} \Omega_A^{n_2-1}(X)$ is a map.

The definition of cohomological products and the pushout properties of $K_{\theta_2\theta_1}$ give a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & K_{\theta_2\theta_1} \arrow[r, "w"] \arrow[d, "t"] & \Omega_A^{n_1+n_2-1}(X) \arrow[r] \arrow[d, "\Omega_A^{n_2-1}(f_{\theta_1})"] & 0 \\
0 \arrow[r] & Y \arrow[r] & K_{\theta_2} \arrow[r, "v"] & \Omega_A^{n_2-1}(X) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Adding F to $K_{\theta_2\theta_1}$ and $\Omega_A^{n_1+n_2-1}(X)$ in the right-most square, we obtain the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega_A^{n_2}(K_{\theta_1}) \arrow[r, equal] \arrow[d] & \Omega_A^{n_2}(K_{\theta_1}) \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & K_{\theta_2\theta_1} \oplus F \arrow[r, "\left(\begin{smallmatrix} w & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "(t,r)"] & \Omega_A^{n_1+n_2-1}(X) \oplus F \arrow[r] \arrow[d, "(\Omega_A^{n_2-1}(f_{\theta_1}),s)"] & 0 \\
0 \arrow[r] & Y \arrow[r] & K_{\theta_2} \arrow[r, "v"] \arrow[d] & \Omega_A^{n_2-1}(X) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in which the right vertical exact sequence is μ and $F \xrightarrow{r} K_{\theta_2}$ is a map with the property that $s = vr$. The left vertical exact sequence is of the form we are seeking. $\square$

We end this section with two results on modules over complete intersections. Recall that a maximal Cohen-Macaulay (or "MCM" from now on) *approximation* of an A-module X is an exact sequence

$$0 \to Y_X \to C_X \to X \to 0$$

where C_X is MCM and Y_X has finite injective dimension, and that a *hull of finite injective dimension* of X is an exact sequence

$$0 \to X \to Y^X \to C^X \to 0$$

where C^X is MCM and Y^X has finite injective dimension. These notions were introduced in [AuB], where it was shown that every finitely generated module over a commutative Noetherian ring admitting a dualizing module has a MCM approximation and a hull of finite injective dimension. The following result provides a simple proof of the complete intersection case, using a technique similar to the proof of the main result in [Bak] and the fact that over a complete intersection every module has reducible complexity.

Proposition 2.4. *Suppose A is a complete intersection.*

(i) *If $\eta \in \operatorname{Ext}_A^{|\eta|}(M,M)$ reduces the complexity of M, then so does η^t for $t \geq 1$.*
(ii) *Every A-module has a MCM approximation and a hull of finite injective dimension.*

Proof. (i) Using Lemma 2.3 it is easily proved by induction on t that $\operatorname{cx} K_{\eta^t} \leq \operatorname{cx} K_\eta$.

(ii) Fix an exact sequence

$$0 \to Y \to C \to M \to 0$$

where C is MCM (the minimal free cover of M, for example). If the complexity of Y is nonzero, let $\eta \in \operatorname{Ext}_A^{|\eta|}(Y,Y)$ be an element reducing it, and choose an integer $t \geq 1$ with the property that $\Omega_A^{t|\eta|-1}(Y)$ is MCM. The element η^t is given by the exact sequence

$$0 \to Y \to K_{\eta^t} \to \Omega_A^{t|\eta|-1}(Y) \to 0,$$

and by (i) it also reduces the complexity of Y. From the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & K_{\eta^t} \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to Y' \to C' \to M \to 0$$

(with $Y' = K_{\eta^t}$), where C' is MCM and $\operatorname{cx} Y' < \operatorname{cx} Y$. Repeating the process we eventually obtain a MCM approximation of M, since over a Gorenstein ring a module has finite injective dimension precisely when it has finite projective dimension.

As for a hull of finite injective dimension, fix an exact sequence

$$0 \to M \to Y \to C \to 0$$

where C is MCM (obtained for example from a suitable power of an element in $\operatorname{Ext}_A^*(M,M)$ reducing the complexity of M). If the complexity of Y is nonzero, choose as above an element $\eta \in \operatorname{Ext}_A^{|\eta|}(Y,Y)$ reducing it, and let $t \geq 1$ be an integer such that $\Omega_A^{t|\eta|-1}(Y)$ is MCM. From the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & K_{\eta^t} \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to M \to Y' \to C' \to 0$$

(with $Y' = K_{\eta^t}$), where C' is MCM and $\operatorname{cx} Y' < \operatorname{cx} Y$. Repeating the process we eventually obtain a hull of finite injective dimension of M. $\square$

3. Vanishing results

This section investigates the vanishing of cohomology and homology for a module having reducible complexity, and *the following is assumed throughout*:

Assumption. The module M has reducible complexity, and N is a nonzero A-module. If $\operatorname{cx} M > 0$, then there exist A-modules $K_1, \dots, K_c$ and a set of cohomological elements $\{\eta_i \in \operatorname{Ext}_A^{|\eta_i|}(K_{i-1}, K_{i-1})\}_{i=1}^{c}$ given by exact sequences

$$0 \to K_{i-1} \to K_i \to \Omega_A^{|\eta_i|-1}(K_{i-1}) \to 0$$

for $i = 1, \dots, c$ (where $K_0 = M$), satisfying $\operatorname{depth} K_i = \operatorname{depth} M, \operatorname{cx} K_i < \operatorname{cx} K_{i-1}$ and $\operatorname{cx} K_c = 0$ (such elements η_i must exist by Definition 2.1).

For an A-module N, we define $\mathrm{q}^A(M,N)$ and $\mathrm{p}^A(M,N)$ by

$$\begin{aligned} \mathrm{q}^A(M,N) &= \sup\{n \mid \operatorname{Tor}_n^A(M,N) \neq 0\}, \\ \mathrm{p}^A(M,N) &= \sup\{n \mid \operatorname{Ext}_A^n(M,N) \neq 0\}. \end{aligned}$$

The definition of modules having reducible complexity suggests that when proving results about $\mathrm{q}^A(M,N)$ and $\mathrm{p}^A(M,N)$, we use induction on the complexity of M.

The first result and its corollary (which considers a conjecture of Auslander and Reiten) consider the vanishing of cohomology, and generalize [ArY, Theorem 4.2 and Theorem 4.3].

Theorem 3.1. *The following are equivalent.*

(i) *There exists an integer* $t > \operatorname{depth} A - \operatorname{depth} M$ *such that* $\operatorname{Ext}_A^{t+i}(M,N) = 0$ *for* $0 \leq i \leq |\eta_1| + \cdots + |\eta_c| - c$.

(ii) $\mathrm{p}^A(M,N) < \infty$.

(iii) $\mathrm{p}^A(M,N) = \operatorname{depth} A - \operatorname{depth} M$.

Proof. We only need to show the implication (i) $\Rightarrow$ (iii), and we do this by induction on $\operatorname{cx} M$. If the projective dimension of M is finite, then by the Auslander-Buchsbaum formula it is equal to $\operatorname{depth} A - \operatorname{depth} M$. Since N is finitely generated, we have $N \neq \mathfrak{m} N$ by Nakayama's Lemma, hence there exists a nonzero element $x \in N \setminus \mathfrak{m} N$. The map $A \to N$ defined by $1 \mapsto x$ then gives rise to a nonzero element of $\operatorname{Ext}_A^{\operatorname{pd} M}(M,N)$, and therefore $\mathrm{p}^A(M,N) = \operatorname{depth} A - \operatorname{depth} M$.

Now suppose the complexity of M is nonzero, and consider the exact sequence

$$(\dagger) \qquad 0 \to M \to K_1 \to \Omega_A^{|\eta_1|-1}(M) \to 0.$$

The vanishing interval for $\operatorname{Ext}_A^i(M,N)$ implies that $\operatorname{Ext}_A^{t+i}(K_1,N) = 0$ for $0 \leq i \leq |\eta_2| + \cdots + |\eta_c| - (c-1)$, and so by induction $\mathrm{p}^A(K_1,N) = \operatorname{depth} A - \operatorname{depth} K_1$. Since we have equalities $\mathrm{p}^A(M,N) = \mathrm{p}^A(K_1,N)$ and $\operatorname{depth} M = \operatorname{depth} K_1$, we are done. □

Corollary 3.2. $\mathrm{p}^A(M,M) = \operatorname{pd} M$.

Proof. Suppose $\mathrm{p}^A(M,M) < \infty$. If the projective dimension of M is not finite, i.e. if $\operatorname{cx} M > 0$, then consider the exact sequence ($\dagger$) representing η_1 (from the proof of Theorem 3.1), where $K_1 = K_{\eta_1}$. Since η_1 is nilpotent there is an integer t such that $\eta_1^t = 0$, and therefore $\operatorname{cx} K_{\eta_1^t} = \operatorname{cx} M$. But using Lemma 2.3 we see that $\operatorname{cx} K_{\eta_1^i} \leq \operatorname{cx} K_{\eta_1}$ for all $i \geq 1$, and since $\operatorname{cx} K_{\eta_1} < \operatorname{cx} M$ we have reached a contradiction. Therefore the projective dimension of M is finite and equal to $\operatorname{depth} A - \operatorname{depth} M$ by the Auslander-Buchsbaum formula, and from Theorem 3.1 we see that $\mathrm{p}^A(M,M) = \operatorname{pd} M$. □

The next result is a homology version of Theorem 3.1, and it is closely related to [Jo1, Theorem 2.1].

Theorem 3.3. *The following are equivalent.*

(i) *There exists an integer* $t > \operatorname{depth} A - \operatorname{depth} M$ *such that* $\operatorname{Tor}_{t+i}^A(M,N) = 0$ *for* $0 \leq i \leq |\eta_1| + \cdots + |\eta_c| - c$.
(ii) $\mathrm{q}^A(M,N) < \infty$.
(iii) $\operatorname{depth} A - \operatorname{depth} M - \operatorname{depth} N \leq \mathrm{q}^A(M,N) \leq \operatorname{depth} A - \operatorname{depth} M$.

Proof. We only need to show the implication (i) $\Rightarrow$ (iii), and we do this by induction on $\operatorname{cx} M$. The case $\operatorname{pd} M < \infty$ follows from the Auslander-Buchsbaum formula and [ChI, Remark 8], so suppose therefore $\operatorname{cx} M > 0$, and consider the exact sequence (†) from the proof of Theorem 3.1. Since $\operatorname{Tor}_{t+i}^A(K_1, N) = 0$ for $0 \leq i \leq |\eta_2| + \cdots + |\eta_c| - (c-1)$, we get by induction that the inequalities hold for K_1 and N. But as in the previous proof we have $\mathrm{q}^A(M,N) = \mathrm{q}^A(K_1,N)$ and $\operatorname{depth} M = \operatorname{depth} K_1$, hence the inequalities hold for M and N. $\square$

The following result contains half of [ChI, Theorem 3] (and a version of the first half of [ArY, Theorem 2.5]) and the main result from [Jo2] for Cohen-Macaulay rings, which says that the integer $\mathrm{q}^A(M,N)$ can be computed locally. It also establishes the *depth formula* provided N is maximal Cohen-Macaulay.

Theorem 3.4. (i) *If* $\mathrm{q}^A(M,N)$ *is finite and* $\operatorname{depth} \operatorname{Tor}_{\mathrm{q}^A(M,N)}^A(M,N) = 0$, *then*

$$\mathrm{q}^A(M,N) = \operatorname{depth} A - \operatorname{depth} M - \operatorname{depth} N.$$

(ii) *If A is Cohen-Macaulay and* $\mathrm{q}^A(M,N)$ *is finite, then the equality*

$$\mathrm{q}^A(M,N) = \sup\{\operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} A\}$$

holds.

(iii) *If* $\mathrm{q}^A(M,N) = 0$ *and N is maximal Cohen-Macaulay, then the depth formula holds for M and N, i.e.*

$$\operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} A + \operatorname{depth}(M \otimes N).$$

Proof. (i) We argue by induction on $\operatorname{cx} M$, the case $\operatorname{pd} M < \infty$ following from [Aus, Theorem 1.2] and the Auslander-Buchsbaum formula. Suppose therefore that the complexity of M is nonzero, and consider the exact sequence (†) from the proof of Theorem 3.1. Since $\operatorname{Tor}_{\mathrm{q}^A(M,N)}^A(M,N)$ is a submodule of $\operatorname{Tor}_{\mathrm{q}^A(M,N)}^A(K_1,N)$, the latter is also of depth zero, hence by induction and the equalities $\mathrm{q}^A(K_1,N) = \mathrm{q}^A(M,N), \operatorname{depth} K_1 = \operatorname{depth} M$ we are done.

(ii) Suppose $\mathrm{q}^A(M,N)$ is finite, and let $\mathfrak{p} \subseteq A$ be a prime ideal. If M has finite projective dimension, so has $M_{\mathfrak{p}}$, and from Theorem 3.3 we get $\mathrm{q}^{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \geq \operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}}$. If $\operatorname{cx} M > 0$, consider the exact sequences

$$0 \to K_{i-1} \to K_i \to \Omega_A^{|\eta_i|-1}(K_{i-1}) \to 0$$

for $i = 1, \dots, c$ (where $K_0 = M$), satisfying $\operatorname{depth} K_i = \operatorname{depth} M, \operatorname{cx} K_i < \operatorname{cx} K_{i-1}$ and $\operatorname{cx} K_c = 0$. Localizing at $\mathfrak{p}$, we see that $\operatorname{depth}(K_i)_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}}$ (as in the remark following Definition 2.1), and that $\mathrm{q}^{A_{\mathfrak{p}}}((K_i)_{\mathfrak{p}}, N_{\mathfrak{p}}) = \mathrm{q}^{A_{\mathfrak{p}}}((K_{i-1})_{\mathfrak{p}}, N_{\mathfrak{p}})$. As K_c has finite projective dimension we get

$$\begin{aligned} \mathrm{q}^{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) &= \mathrm{q}^{A_{\mathfrak{p}}}((K_c)_{\mathfrak{p}}, N_{\mathfrak{p}}) \\ &\geq \operatorname{ht} \mathfrak{p} - \operatorname{depth}(K_c)_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}} \\ &= \operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}}, \end{aligned}$$

hence since $\mathrm{q}^A(M,N) \geq \mathrm{q}^{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ the inequality

$$\mathrm{q}^A(M,N) \geq \sup\{\operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} A\}$$

holds.

For the reverse inequality, choose any associated prime $\mathfrak{p}$ of $\operatorname{Tor}^A_{\mathrm{q}^A(M,N)}(M,N)$. Then $\mathrm{q}^A(M,N) = \mathrm{q}^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ and $\operatorname{depth} \operatorname{Tor}^{A_\mathfrak{p}}_{\mathrm{q}^{A_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$, and a small adjustment of the proof of (i) above gives

$$\mathrm{q}^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{ht} \mathfrak{p} - \operatorname{depth} M_\mathfrak{p} - \operatorname{depth} N_\mathfrak{p}.$$

(iii) Again we argue by induction on $\operatorname{cx} M$, where the case $\operatorname{pd} M < \infty$ follows from [Aus, Theorem 1.2]. Suppose $\operatorname{cx} M > 0$, and consider the exact sequence (†). We have $\mathrm{q}^A(K_1, N) = 0$, hence by induction the depth formula holds for K_1 and N. Since $\operatorname{depth} K_1 = \operatorname{depth} M$, we only have to prove that the equality $\operatorname{depth}(M \otimes N) = \operatorname{depth}(K_1 \otimes N)$ holds. For each $i \geq 0$ we have an exact sequence

$$0 \to \Omega^{i+1}_A(M) \otimes N \to F_i \otimes N \to \Omega^i_A(M) \otimes N \to 0,$$

and as N is maximal Cohen-Macaulay we must have that $\operatorname{depth}\left(\Omega^i_A(M) \otimes N\right)$ is at most $\operatorname{depth}\left(\Omega^{i+1}_A(M) \otimes N\right)$. In particular the inequality $\operatorname{depth}(M \otimes N) \leq \operatorname{depth}\left(\Omega^{|\eta_1|-1}_A(M) \otimes N\right)$ holds, and therefore when tensoring the sequence (†) with N we see that $\operatorname{depth}(M \otimes N) = \operatorname{depth}(K_1 \otimes N)$. □

The next result deals with symmetry in the vanishing of Ext. It was shown in [AvB] that if X and Y are modules over a complete intersection A, then $\operatorname{Ext}^i_A(X,Y) = 0$ for $i \gg 0$ if and only if $\operatorname{Ext}^i_A(Y,X) = 0$ for $i \gg 0$. This was generalized in [HuJ] to a class of local Gorenstein rings named “AB rings”, a class properly containing the class of complete intersections. Another generalization appeared in [Jør], where techniques from the theory of derived categories were used to show that symmetry in the vanishing of Ext holds for modules of finite complete intersection dimension over local Gorenstein rings.

Theorem 3.5. *If A is Gorenstein then the implication*

$$\mathrm{p}^A(N,M) < \infty \Rightarrow \mathrm{p}^A(M,N) < \infty$$

holds. In particular, symmetry in the vanishing of Ext *holds for modules with reducible complexity over a local Gorenstein ring.*

Proof. Define the integer w (depending on M) by

$$w = \operatorname{depth} A - \operatorname{depth} M + |\eta_1| + \cdots + |\eta_c| - c.$$

If for some integer $i \geq 1$ we have $\operatorname{Tor}^A_i(M,N) = \cdots = \operatorname{Tor}^A_{i+w}(M,N) = 0$, then $\operatorname{Tor}^A_i(M,N) = 0$ for all $i \geq 1$ by Theorem 3.3. The result now follows from [Jør, Theorem 1.7 and Proposition 2.2]. □

The final result deals with the vanishing of homology for two modules when A is a complete intersection. Namely, in this situation the homology modules are given as the homology modules of two modules of finite projective dimension, due to the fact that every module over a complete intersection has reducible complexity.

Theorem 3.6. *Suppose A is a complete intersection, and let X and Y be A-modules such that $\operatorname{Tor}^A_i(X,Y) = 0$ for $i \gg 0$. Then there exist A-modules X' and Y', both of finite projective dimension, such that $\operatorname{depth} X = \operatorname{depth} X', \operatorname{depth} Y = \operatorname{depth} Y'$ and $\operatorname{Tor}^A_i(X,Y) \simeq \operatorname{Tor}^A_i(X',Y')$ for $i > 0$.*

Proof. If the complexity of one of X and Y, say X, is nonzero, choose a homogeneous element $\eta \in \operatorname{Ext}^*_A(X,X)$ reducing the complexity. By Proposition 2.4(i) any power of η also reduces the complexity of X, so choose an integer t such that $\operatorname{Tor}^A_i(\Omega^{t|\eta|-1}_A(X), Y) = 0$ for $i > 0$. The element η^t is given by the short exact sequence

$$0 \to X \to K_{\eta^t} \to \Omega^{t|\eta|-1}_A(X) \to 0,$$

therefore by the choice of t we see that $\operatorname{Tor}_i^A(X,Y)$ and $\operatorname{Tor}_i^A(K_{\eta^t},Y)$ are isomorphic for $i>0$. Since A is Cohen-Macaulay we automatically have $\operatorname{depth} X = \operatorname{depth} K_{\eta^t}$, and repeating this process we eventually obtain what we want. $\square$

This result has consequences for the study of the rigidity of Tor over Noetherian local rings. This study was initiated by M. Auslander in his 1961 paper [Aus], in which he proved his famous rigidity theorem; if X and Y are modules over an unramified regular local ring R, and $\operatorname{Tor}_n^R(X,Y)=0$ for some $n \geq 1$, then $\operatorname{Tor}_i^R(X,Y)=0$ for all $i \geq n$ (recall that a regular local ring $(S,\mathfrak{m}_S)$ is said to be *ramified* if it is of characteristic zero while its residue class field has characteristic $p>0$ and p is an element of $\mathfrak{m}_S^2$). In 1966 S. Lichtenbaum extended Auslander's rigidity theorem to all regular local rings (see [Lic]), and subsequently Peskine and Szpiro conjectured in [PeS] that the theorem holds for all Noetherian local rings provided one of the modules in question has finite projective dimension. A counterexample to the conjecture was provided by Heitmann in [Hei], where a Cohen-Macaulay ring R together with R-modules X and Y were given, for which $\operatorname{pd} X = 2$ and $\operatorname{Tor}_1^R(X,Y)=0$, while $\operatorname{Tor}_2^R(X,Y) \neq 0$.

However, whether the rigidity of Tor holds for Noetherian local rings provided *both* modules involved have finite projective dimension is unknown. If this holds over complete intersections, then Theorem 3.6 shows that the conjecture of Peskine and Szpiro also holds for such rings (i.e. rigidity of Tor holds provided one of the modules involved has finite projective dimension). In fact, the theorem shows that if rigidity holds over a complete intersection R provided both modules have finite projective dimension, then rigidity holds over R for all modules X and Y satisfying $\operatorname{Tor}_i^R(X,Y)=0$ for $i \gg 0$.

4. A generalization

In this final section we discuss a situation which slightly generalizes the concept of reducible complexity. Instead of letting M have reducible complexity as in Definition 2.1, we make the following assumption:

Assumption. The complexity of M is finite, and if it is nonzero then there exist local rings $\{R_i\}_{i=1}^c$ such that for each $i \in \{1,\dots,c\}$ there is a faithfully flat local homomorphism $R_{i-1} \to R_i$ (where $R_0 = A$), an R_i-module K_i, an integer n_i and an exact sequence

$$0 \to R_i \otimes_{R_{i-1}} K_{i-1} \to K_i \to \Omega_{R_i}^{n_i}(R_i \otimes_{R_{i-1}} K_{i-1}) \to 0$$

(where $K_0 = M$) satisfying $\operatorname{depth}_{R_i} K_i = \operatorname{depth}_{R_i}(R_i \otimes_{R_{i-1}} K_{i-1}), \operatorname{cx}_{R_i} K_i < \operatorname{cx}_{R_{i-1}} K_{i-1}$ and $\operatorname{pd}_{R_c} K_c < \infty$.

Of course, if M has reducible complexity then by choosing each R_i to be A we see that the assumption is satisfied. Now let $S \to T$ be any faithfully flat local homomorphism, and X and Y any (finitely generated) S-modules. If $\mathbf{F}_X$ is a minimal S-free resolution of X, then the complex $T \otimes_S \mathbf{F}_X$ is a minimal T-free resolution of $T \otimes_S X$, and by [EGA, Proposition (2.5.8)] we have natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_T(T \otimes_S \mathbf{F}_X, T \otimes_S Y) &\simeq T \otimes_S \operatorname{Hom}_S(\mathbf{F}_X, Y), \\ (T \otimes_S \mathbf{F}_X) \otimes_T (T \otimes_S Y) &\simeq T \otimes_S (\mathbf{F}_X \otimes_S Y). \end{aligned}$$

Therefore we have isomorphisms

$$\begin{aligned} \operatorname{Ext}_T^i(T \otimes_S X, T \otimes_S Y) &\simeq T \otimes_S \operatorname{Ext}_S^i(X,Y), \\ \operatorname{Tor}_i^T(T \otimes_S X, T \otimes_S Y) &\simeq T \otimes_S \operatorname{Tor}_i^S(X,Y), \end{aligned}$$

and as T is faithfully S-flat we then get

$$\begin{aligned}
\operatorname{Ext}^i_T(T \otimes_S X, T \otimes_S Y) = 0 &\quad \Leftrightarrow \quad \operatorname{Ext}^i_S(X,Y) = 0, \\
\operatorname{Tor}_i^T(T \otimes_S X, T \otimes_S Y) = 0 &\quad \Leftrightarrow \quad \operatorname{Tor}_i^S(X,Y) = 0.
\end{aligned}$$

We then get the equalities

$$\begin{aligned}
\operatorname{cx}_S X &= \operatorname{cx}_T(T \otimes_S X) \\
\mathrm{p}^S(X,Y) &= \mathrm{p}^T(T \otimes_S X, T \otimes_S Y) \\
\mathrm{q}^S(X,Y) &= \mathrm{q}^T(T \otimes_S X, T \otimes_S Y) \\
\operatorname{depth}_S X - \operatorname{depth}_S Y &= \operatorname{depth}_T(T \otimes_S X) - \operatorname{depth}_T(T \otimes_S Y),
\end{aligned}$$

where the one involving depth follows from [Mat, Theorem 23.3].

Using the above facts it is easy to see that both Theorem 3.1 and Corollary 3.2 remain true in this new situation, as does Theorem 3.3 if we drop the left inequality in (iii).

Suppose now that M has finite complete intersection dimension. Then [AGP, Proposition 7.2] and an argument similar to the proof of Proposition 2.2(i) show that M satisfies this new assumption *and that* $n_i = 1$ *for each* $1 \leq i \leq c$. Consequently, the vanishing intervals in Theorem 3.1(i) and Theorem 3.3(i) are of length $\operatorname{cx}_A M + 1$, as in [AvB, Theorem 4.7] and [Jo1, Theorem 2.1]. Moreover, we obtain [AvB, Theorem 4.2], which says that M is of finite projective dimension if and only if $\operatorname{Ext}^{2n}_A(M,M) = 0$ for some $n \geq 1$. To see this, note that when $\operatorname{cx}_A M > 0$ the extension

$$0 \to R_1 \otimes_A M \to K_1 \to \Omega^1_{R_1}(R_1 \otimes_A M) \to 0$$

corresponds to an element $\theta \in \operatorname{Ext}^2_{R_1}(R_1 \otimes_A M, R_1 \otimes_A M)$. If $\operatorname{Ext}^{2n}_A(M,M) = 0$ for some $n \geq 1$, then $\operatorname{Ext}^{2n}_{R_1}(R_1 \otimes_A M, R_1 \otimes_A M)$ also vanishes, hence $\theta^{2n} = 0$. As in the proof of Corollary 3.2 we obtain the contradiction

$$\begin{aligned}
\operatorname{cx}_A M &= \operatorname{cx}_{R_1}(R_1 \otimes_A M) \\
&= \operatorname{cx}_{R_1} K_{\theta^{2n}} \\
&\leq \operatorname{cx}_{R_1} K_1 \\
&< \operatorname{cx}_A M,
\end{aligned}$$

showing that we cannot have $\operatorname{Ext}^{2n}_A(M,M) = 0$ for some $n \geq 1$ when M is of positive complexity.

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